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LETTER TO THE EDITOR

Compact directed percolation with modified boundary rules: a forest fire model

Michael J Kearney

Department of Electronic and Electrical Engineering, Loughborough University, Loughborough, Leicestershire, LE11 3TU, UK

E-mail: m.j.kearney@lboro.ac.uk

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Abstract
A modified version of compact directed percolation on a square lattice is examined in the context of a model problem for the spread of forest fires. The modification relates to conditioning the extent of the fire spread along the forest boundaries. Exact expressions are given for the mean perimeter length and the mean size of the damaged forest under such conditioning.

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Compact directed percolation (CDP) on a square lattice has been widely studied via its links to the Domany–Kinzel cellular automaton [1] and by direct cluster enumeration [2, 3]. The fact that CDP can be solved exactly makes it interesting from the perspective of studying the spread of forest fires, epidemics and avalanches [4, 5]. The prospect of finding exact solutions also provides a motivation for analysing variants of the basic model [6–10]. One such variant is considered here, developed in the context of a simple forest fire model.

Consider a forest defined on a square lattice with tree locations labelled by integer pairs (x_1, x_2), where $x_1, x_2 \geqslant 0$ (figure 1). At time $t = 0$, the tree at the origin is burning with probability 1. At integer time t, the state of every tree on a shell, whose chemical distance from the origin $x_1 + x_2 = t$, is determined depending upon the state of the nearest neighbour trees in the preceding shell. For trees interior to the forest (with $x_1 \neq 0$ and $x_2 \neq 0$) there are two such neighbours and the conditional probability rules are as for CDP: $P[1\,|\,0, 0] = 0$, $P[1\,|\,1, 0] = P[1\,|\,0, 1] = p$ and $P[1\,|\,1, 1] = 1$, where 1(0) corresponds to burning (not burning). However, for trees on the forest boundaries (the x_1 and x_2 axes) we now construct a modified rule, wherein the first m and n trees respectively catch fire with probability 1, whereafter all such trees remain intact (see figure 1). This allows the 'scale' of the resulting fire damage to be investigated with respect to the boundary rule. This particular rule has been chosen for ease of presentation of the key ideas. More complicated examples of boundary rules that can also be handled using the same methods are given at the end.

The clusters of burned trees are fully compact (i.e. contain no holes) and the model exhibits a phase transition at $p = p_c = 1/2$ (see below). Finite clusters may be characterized by the

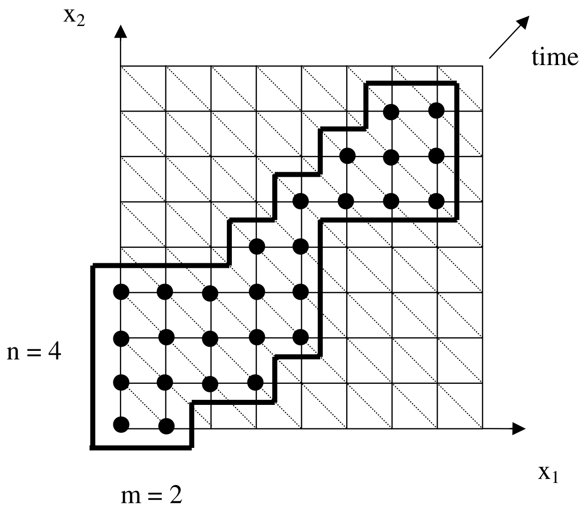

Figure 1. A typical fire-damaged cluster of trees generated in 14 time-steps, with *given* values of $m = 2$ and $n = 4$. The diagonal lines link sites on a shell. The circles represent damaged trees, and the solid line is the perimeter staircase polygon. For this cluster, $\ell = 32$, $s = 27$ and the cluster weighting is $p^{10}q^{14}$.

low-order moments of their perimeter length and size. The perimeter length is defined to be the length of the perimeter of the so-called 'staircase' polygon [3, 10–12] (defined on the dual lattice) that bounds the cluster as tightly as possible (figure 1). The cluster size is simply the number of burned trees or the area of the staircase polygon. To evaluate the relevant moments, we consider the area-perimeter generating function $g_{mn}(y, z)$ for staircase polygons defined for *given* values m and n (see figure 1). By definition,

$$g_{mn}(y, z) \equiv \sum_{\ell,s} C^{mn}_{\ell s} y^{\ell} z^{s} \tag{1}$$

where y is the perimeter 'activity', z is the area 'activity', and $C^{mn}_{\ell s}$ is the number of compact clusters of perimeter length ℓ and area s restricted to have given values of m and n. Adapting the arguments in [2, 10] concerning the properties of random walks, the probabilistic weight of a restricted cluster of perimeter length ℓ is given by $p^{-(m+n)}q^{-2}(pq)^{\ell/2}$, where $q = 1 - p$. This, together with (1), means that the probability, Q_{mn}, that the fire eventually dies out is given by

$$Q_{mn} \equiv p^{-(m+n)}q^{-2}g_{mn}(\sqrt{pq}, 1). \tag{2}$$

Further, the mean perimeter length, L_{mn}, and the mean size, S_{mn}, of the damaged area (*given* that the damage remains finite, which introduces a factor Q^{-1}_{mn}) are given by

$$L_{mn} = \langle \ell \rangle_{mn} \equiv Q^{-1}_{mn} p^{-(m+n)} q^{-2} \left(y \frac{\partial g_{mn}(y, 1)}{\partial y} \right)\bigg|_{y=\sqrt{pq}} \tag{3}$$

$$S_{mn} = \langle s \rangle_{mn} \equiv Q^{-1}_{mn} p^{-(m+n)} q^{-2} \left(\frac{\partial g_{mn}(\sqrt{pq}, z)}{\partial z} \right)\bigg|_{z=1}. \tag{4}$$

It follows from a duality argument that (3) and (4) are symmetric about $p = p_c$ [2]. In what follows, we mainly consider the regime $p < p_c$ when the fire damage remains finite with probability 1; see below. The task reduces to evaluating the area-perimeter generating function and its derivatives at $y = \sqrt{pq}$ and $z = 1$. The main purpose of this letter is to show how to evaluate (2), (3) and (4) exactly.

It was shown in [10] in the context of another modified CDP problem that $g_{mn}(y, z)$ obeys the following recursion relation:

$$zg_{m+1,n} = y^2 z^{n+1} g_{m,n} + g_{m+1,n+1} \qquad m, n \geqslant 1. \tag{5}$$

This recursion has not yet been solved for arbitrary y, z although related generating functions have been found explicitly in terms of q-series [11, 12]. However, when $z = 1$ it follows from (5) that

$$\begin{aligned} g_{mn}(y, 1) &= y^4 \lambda^{m+n-2} \\ \lambda(y) &= \frac{1 - \sqrt{1 - 4y^2}}{2}. \end{aligned} \tag{6}$$

This expression is sufficient to evaluate (2) and (3). For $p < p_c = 1/2$ we have $Q_{mn} = 1$ and for $p > p_c$ we have $Q_{mn} = (q/p)^{m+n-2} < 1$ (except $Q_{11} = 1$). Thus the fire definitely stops spreading if $p < p_c$, but may continue indefinitely if $p > p_c$ with probability $P_\infty = 1 - Q_{mn}$. For $p \to p_c{}^+$ we have $P_\infty \sim 4(m + n - 2)(p - p_c)$ with exponent $\beta = 1$. For $p < p_c$ the mean perimeter length of the fire-damaged area is given by

$$L_{mn} = 4 + 2(m + n - 2) \left(\frac{1 - p}{1 - 2p} \right) \tag{7}$$

and this diverges as $p \to p_c{}^-$ with exponent $\tau = 1$. Asymptotically, we have

$$L_{mn} \sim \left(\frac{m + n - 2}{2} \right) \frac{1}{(p_c - p)}. \tag{8}$$

Equivalently, since $\ell = 2t + 4$ where t is the time taken by a particular fire to die out, we can consider the mean time, T_{mn}, until the fire stops spreading,

$$T_{mn} \equiv \langle t \rangle_{mn} = (m + n - 2) \left(\frac{1 - p}{1 - 2p} \right).$$

Calculating the mean size of the damaged area is significantly more difficult, as one needs information about the derivative of g_{mn} with respect to z (see (4)). Differentiating (5) directly and setting $z = 1$ gives

$$f_{m+1,n} - y^2 f_{mn} - f_{m+1,n+1} = (n + 1) y^6 (\lambda)^{m+n-2} - y^4 (\lambda)^{m+n-1} \tag{9}$$

where $f_{mn}(y) \equiv \partial g_{mn}(y, z)/\partial z|_{z=1}$. The solution of this non-trivial inhomogeneous recursion may be found by assuming a (symmetric) trial solution of the form

$$f_{mn}(y) = [A(m^2 + n^2) + B(m + n) + Cmn + D] \lambda^{m+n} \tag{10}$$

where A, B, C and D are functions of y, λ. After some straightforward algebra these coefficients can be evaluated (see [10] for details), whereupon the mean cluster size for $p < p_c$ is given by

$$\begin{aligned} S_{mn} = \Bigg[&\frac{1}{2(1 - 2p)} (p(1 - p)(m^2 + n^2) - p(m + n) + 2(1 - p)^2 mn) \\ &+ \frac{p^2}{2(1 - 2p)^2} (m + n - 2) \Bigg]. \end{aligned} \tag{11}$$

This expression diverges as $p \to p_c{}^-$ with exponent $\gamma = 2$ with the following asymptotic behaviour:

$$S_{mn} \sim \frac{m + n - 2}{32(p_c - p)^2}. \tag{12}$$

Note from (7) and (11) that for $m = n = 1$ we have $L_{11} = 4$ and $S_{11} = 1$. This is expected since if the fire cannot spread along the forest boundaries then no other trees (apart from the original tree) can catch fire. Also, when $p = 0$ we have $L_{mn} = 2m + 2n$ and $S_{mn} = mn$; this is again expected since a rectangular area of damage is guaranteed as a minimum by the compactness rule $P[1 \mid 1, 1] = 1$.

Having derived (7) and (11) it is interesting to compare them with the results obtained using the conventional CDP boundary rule wherein the boundary trees catch fire with probability p conditional on their predecessor catching fire [2, 10]:

$$L = 4\left(\frac{1-p}{1-2p}\right) \sim \frac{1}{p_c - p} \tag{13}$$

$$S = \left(\frac{1-p}{1-2p}\right)^2 \sim \frac{1}{16(p_c - p)^2}. \tag{14}$$

In each case, the exponents are the same (since the universality class is the same) but the amplitudes are different. If $m + n \geqslant 4$ then $L_{mn} > L$ and $S_{mn} > S$ for all values of p. In this situation, the spread of the fire will be greater on average than would be the case if the boundary trees were able to burn with probability p (conditional on their predecessor). If $m + n < 4$ the converse can occur as is clearly evident by considering the dominant diverging terms in (8), (12), (13) and (14). Note that the ratio $S_{mn}/L_{mn} \sim S/L \sim (16(p_c - p))^{-1}$; in other words the amplitudes L_{mn} and S_{mn} scale asymptotically by the same factor of $(m + n - 2)/2$. The same holds true for P_∞.

Approaching the critical point the fluctuations become large. Some measure of this can be gained by considering the variance of the perimeter length distribution, which can be deduced using (6) and an obvious generalization of (3). The result is complicated, but (asymptotically) we have

$$\langle \ell^2 \rangle_{mn} \sim \frac{m + n - 2}{8(p_c - p)^3} \tag{15}$$

so the variance diverges with exponent $\theta = 3$. Note that asymptotically all moments have an amplitude $\propto (m + n - 2)$.

There are several generalizations of the above that can be handled relatively easily using the given results (although the resulting expressions are very complicated which is why they are not given here). For example, one can consider the case where trees on the boundary burn with some fixed probability (not necessarily p) conditional on their predecessor, but only up to some fixed values of m and n (a 'fire-break' model). Or one can assume that the first m and n boundary trees burn with probability 1 and thereafter burn with some other fixed probability conditional on their predecessor (a 'flame-front' model). The solution for such boundary rules requires only a simple redefining of the cluster weighting and the summation of various geometric series.

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